

Comment on “Quantization of Diffeomorphism-Invariant Theories with Fermions”

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Abstract

In the comment to the article by J. Baez and K. Krasnov (hep-th/9703112) are discussed some topics related with application of certain constructions to non-trivial principal bundles.

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In chapter 2 of the article [1] is used following construction:

(...) First, define a ‘transporter’ from the point p to the point q to be a map from P_p to P_q that commutes with the right action of G on the bundle P . If we trivialize the bundle over p and q , we can think of such a transporter simply as an element of G . A ‘generalized connection’ A is a map assigning to each oriented analytic path e in Σ a parallel transporter $A_e: P_p \rightarrow P_q$, where p is the initial point of the path e and q is the final point. We require that A satisfy certain obvious consistency conditions: A should assign the same transporter to two paths that differ only by an orientation-preserving reparametrization, it should assign to the inverse of any path the inverse transporter, and it should assign to the composite of two paths the composite transporter.

An ordinary smooth connection A gives a generalized connection where the parallel transporter A_e along any path e is simply the holonomy of A along this path (...)

The *trivialization* here is identification of fibers P_p and P_q with G (structure group) in the both points p and q , so the ‘transporter’ can be represented as left action of G on G .

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$$\begin{array}{ccc}
P_p & \xrightarrow{\quad} & P_q \\
& \searrow \quad \swarrow & \\
& G &
\end{array} \tag{1}$$

Let us consider a connection on the principal bundle $P(M, G)$ with total space P , structure group G and base space M . P_x is fiber in $x \in M$. It should be emphasized that for open path it is necessary to consider an additional structure, a trivialization at endpoints, while for closed path there is a way to introduce holonomy group without using a trivialization. For closed path the holonomy group has well defined “internal” description as a group of automorphisms of a fiber and a choice of a point in the fiber produces unique homomorphism to the group G [2]. For open path γ the parallel transport along given path also produces unique image of a point $u_p \in P_p$, it is ending point $u_q \in P_q$ of horizontal lift of γ with initial point u_q , but it is the points in *different* fibers. To find an element of G for the point u_p above p and the point u_q above q it is necessary to map both fibers to group G as in diagram (1).¹

It is always possible to find maps (1) for two given points of base space M , but if it is necessary to consider *any path* on M , the points p and q can be any points of M .

Is it possible to identify a fiber P_x in any point $x \in M$ with group G ? There is simple proof that such *construction maybe continuous only for trivial principal bundle*: let us identify fiber P_x with group G in each point of M . Then for each point of M it is possible to choose point of P_x that corresponds to unit $(\mathbb{I}_G)_x$ of group G . It produces *section* of principal bundle, but only trivial principal bundles $P = M \times G$ can have global section [2, 3]. For non-trivial principal bundle the fiber is equivalent with G only as with topological space without given structure of a group.

Let us consider the trivial principal bundle. It is possible to consider any fiber as group G and the global trivialization produces a general method to calculation of cylinder function [1]

$$\Psi(A) = \psi(\mathcal{P} \exp \int_{\gamma_1} A, \dots, \mathcal{P} \exp \int_{\gamma_n} A)$$

¹There is other method for a given map $t_{p,q}: P_p \rightarrow P_q$. Then it is possible to work with elements m_q and $t_{p,q}(m_p)$ of the same fiber. It will be discussed further.

for analytic paths $\gamma_1, \dots, \gamma_n$ in M with arbitrary endpoints. The example is mentioned here, because only using of the trivialization, an auxiliary structure in the model, make possible to calculate value of the integrals above. Let us compare the construction with some other gauge theories. In the theories together with principal bundle of gauge field there is an associated bundle, for example matter fields and any open path defines action of group G on section F of the associated bundle $E(M, F, G, P)$.

The comparison justify consideration of trivialization as some analogue of the additional structure for more straightforward work with integral above. It is possible, because any trivial principal bundle defines unique *canonical flat connection* with zero curvature [2]. The second connection make possible to consider element $g \in G$ by comparison of two horizontal lifts of the same path by two different connections.

The construction with two connections also works for non-trivial principal bundle. Second connection here could not be treated as some trivialization, but it make possible to calculate element of G for any open path γ and given initial point $u \in P$. Instead of two different connections on the same principal bundle it is possible to consider a second connection on associated bundle $E(M, F=G, G, P)$ with a fiber is the same group G . Here we have two different bundles with the same structure group, and it make the construction similar to “traditional” gauge theories discussed earlier.

The construction could be considered as some abstract exercise without relation with article under consideration, but let us consider concrete example of application of the method.

The $SU(2)$ group of spin network appears as $Spin(3)$, — double cover of $SO(3)$ group of 3D space rotations. Let us recall direct construction of the spin group with using of Clifford algebra [4, 5].

The Clifford algebra $\mathfrak{U}_{0,3}$ is isomorphic [6] with $\mathbb{H} \oplus \mathbb{H}$. Here \mathbb{H} is algebra of quaternions and element of $\mathfrak{U}_{0,3}$ can be represented as algebra of matrices:

$$\mathbf{a} \oplus \mathbf{b} \equiv \begin{bmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{b} \end{bmatrix} \quad \mathbf{a}, \mathbf{b} \in \mathbb{H}$$

Three generators of the algebra $\mathfrak{U}_{0,3}$ may be chosen as $\mathbf{e}_1 = (-\mathbf{i}) \oplus \mathbf{i}'$, $\mathbf{e}_2 = (-\mathbf{j}) \oplus \mathbf{j}'$, $\mathbf{e}_3 = (-\mathbf{k}) \oplus \mathbf{k}'$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ are three quaternionic units $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ of first and second terms of direct sum. Then $\mathbf{e}_n \mathbf{e}_m = -\mathbf{e}_m \mathbf{e}_n$, $m \neq n$, $\mathbf{e}_n^2 = -\mathbf{1} \equiv (-1) \oplus (-1)$, $m, n = 1, 2, 3$.

A group $\mathfrak{U}_{0,3}^*$ of all invertible elements of $\mathfrak{U}_{0,3}$ can be also written with

multiplicative notation as $\mathbb{H}_0 \times \mathbb{H}_0$, where $\mathbb{H}_0 \equiv \mathbb{H} - \{0\}$. A $pin(3)$ is a group generated by multiplication of arbitrary number of elements $\sum a_n \mathbf{e}_n$, $\sum a_n^2 = 1$ and a $Spin(3)$ subgroup is composed by only even number of such terms. So, the $Spin(3)$ group could be written as $\sum c_\alpha e_\alpha$, $\sum c_\alpha^2 = 1$ where e_α are *four* different even combinations of \mathbf{e}_n , *i.e.*: $e_0 \equiv \mathbf{1} = -\mathbf{e}_n^2 = 1 \oplus 1$, $e_1 \equiv -\mathbf{e}_2 \mathbf{e}_3 = \mathbf{i} \oplus \mathbf{i}'$, $e_2 \equiv -\mathbf{e}_3 \mathbf{e}_1 = \mathbf{j} \oplus \mathbf{j}'$, $e_3 \equiv -\mathbf{e}_1 \mathbf{e}_2 = \mathbf{j} \oplus \mathbf{j}'$. The subgroup is isomorphic with $SU(2)$ and we have following structure of the group $\mathfrak{U}_{0,3}^*$: $\mathbb{H}_0 \times \mathbb{H}_0 = \mathbb{R}_+ \times SU(2) \times \mathbb{R}_+ \times SU(2)$, where \mathbb{R}_+ is multiplicative group of positive real numbers.

Let us now consider the Clifford algebra bundle $\mathfrak{U}_{0,3}$ on a manifold. It is possible [5], but already existence of $\mathfrak{U}_{0,3}^*$ bundles is not guaranteed globally. Let us suppose for simplicity that there the $\mathfrak{U}_{0,3}^*$ principal bundle is exist and look for conditions for $Spin(3)$ bundle.

A reduction of principle bundle with structure group G to closed subgroup H is possible if and only if associated bundle with fiber G/H accepts a global section [2] . It is possible for particular case of G/H is diffeomorphic to Euclidean space [2] , like the subgroups $\mathbb{R}_+ \sim \mathbb{R}$, so the reduction of $\mathfrak{U}_{0,3}^*$ to $SU(2) \times SU(2)$ subgroup is always possible.

It is possible now to work with the $SU(2) \times SU(2) \cong Spin(4)$ group² instead of $\mathfrak{U}_{0,3}^*$ and to consider a reduction of the group to $SU(2) \cong Spin(3)$. The condition of the reduction is existence of a section of associated bundle with fiber G/H , *i.e.* the same group $SU(2) \cong Spin(4)/Spin(3)$.

Summarizing the example : If it is possible to make reduction of $\mathfrak{U}_{0,3}^*$ multiplicative group of Clifford bundle on manifold if and only if associated bundle with fiber $SU(2)$ accepts a global section and if it is possible we have two different bundles with the same structure group $SU(2)$: $Spin(3)$ subgroup of initial $\mathfrak{U}_{0,3}^*$ (or $Spin(4)$) bundle and quotient group $SU(2) \cong Spin(4)/SU(2)$ of associated bundle. The associated bundle has global section, but principal $SU(2)$ bundle, reduction of initial $\mathfrak{U}_{0,3}^*$ or $Spin(4)$ bundle may be either trivial or not, *i.e.* it may have no global section.

Conclusion The example has certain analogies with some properties of construction was used in model was discussed earlier.

²The isomorphism with $Spin(4)$ make the consideration relevant also to some theories with Euclidean gravity.

References

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